

The perihelion of Mercury advance and the light bending calculated in (enhanced) Newton's theory

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Abstract. We show that results of a simple dynamical *gedanken* experiment interpreted according to standard Newton's gravitational theory, may reveal that three-dimensional space is curved. The experiment may be used to reconstruct the curved geometry of space, i.e. its non-Euclidean metric ${}^3g_{ik}$. The perihelion of Mercury advance and the light bending calculated from the Poisson equation ${}^3g^{ik}\nabla_i\nabla_k\Phi = -4\pi G\rho$ and the equation of motion $F^i = ma^i$ in the curved geometry ${}^3g_{ik}$ have the correct (observed) values. Independently, we also show that Newtonian gravity theory may be enhanced to incorporate the curvature of three dimensional space by adding an extra equation which links the Ricci scalar 3R with the density of matter ρ . Like in Einstein's general relativity, matter is the source of curvature. In the spherically symmetric (vacuum) case, the metric of space ${}^3g_{ik}$ that follows from this extra equation agrees, to the expected accuracy, with the metric measured by the Newtonian *gedanken* experiment mentioned above.

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1. Introduction

Newton's theory of gravity was formulated in a flat, Euclidean 3-D space but its basic laws, i.e. the Poisson equation and the equation of motion,

$${}^3g^{ik} \nabla_i \nabla_k \Phi = -4\pi G\rho, \quad (1)$$

$$F_i = ma_i, \quad (2)$$

make perfect sense in a 3-D space with an arbitrary geometry ${}^3g_{ik}$. Indeed, the curvature of space is (potentially) present in Newton's theory. It is easy to argue that the “centrifugal” acceleration of a particle moving with velocity V on a circular orbit equals $a_c = V^2/\mathcal{R}$, and the “gravitational” acceleration in the gravitational field of a spherically symmetric body with the mass \mathcal{M} equals $a_G = G\mathcal{M}/\tilde{r}^2$, with \mathcal{R} being the curvature radius of the circle, and \tilde{r} being its circumferential radius. In flat, i.e. Euclidean, 3-D space these two radii are equal, $\mathcal{R} = \tilde{r}$, but in a space with a non-zero gaussian curvature \mathcal{G} , they are different, $\mathcal{R} \neq \tilde{r}$. Therefore, by measuring centrifugal and gravitational accelerations one may independently measure \mathcal{R} and \tilde{r} , and thus *experimentally* find whether the space is flat (Euclidean) or it has a non-zero Gaussian curvature $\mathcal{G} \neq 0$. Based on that, Abramowicz has recently suggested in [1] that a Newtonian physicist could experimentally determine the metric ${}^3g_{ik}$ of the real physical 3-D space and calculate, according to (1) and (2), the perihelion of Mercury advance and the light bending effects. In this paper we follow this suggestion and calculate both effects within Newton's theory. Surprisingly, the values of the perihelion advance and the light bending agree (to the expected order of M/r) with predictions of Einstein's theory. Here M is the “geometrical” mass of the spherical gravitating body expressed in the convenient “geometrical” units $G = 1 = c$. It is connected to the mass \mathcal{M} expressed in the standard units by $M = G\mathcal{M}/c^2$ and has the dimension of length.

Another point discussed in this paper is based on the following two remarks: (i) Obviously, Newton's gravity theory is a limit of Einstein's general theory of relativity. Should the limit *necessarily* correspond to $\mathcal{G} = 0$? Perhaps not, because Newtonian physicists could discover *within* Newton's theory that $\mathcal{G} \neq 0$. (ii) They could also discover that the curvature of space depends on the distance from the gravity center, $\mathcal{G} = \mathcal{G}(r)$. This would suggest to them, again *within* the framework of Newton's theory, that gravity and curvature are not independent, but instead they are somehow linked. Here we suggest that it is possible to establish the link within an “enhanced” version of Newton's theory, by adding to its standard version defined by (1) and (2) an extra equation,

$${}^3R = 2k\rho, \quad (3)$$

where 3R is the Ricci scalar corresponding to ${}^3g_{ik}$, ρ is the density of matter, and k is a constant. Equations (1), (2) and (3) define our enhanced version of Newtonian gravitational theory. In the special case of a spherically symmetric, vacuum ($\rho = 0$)

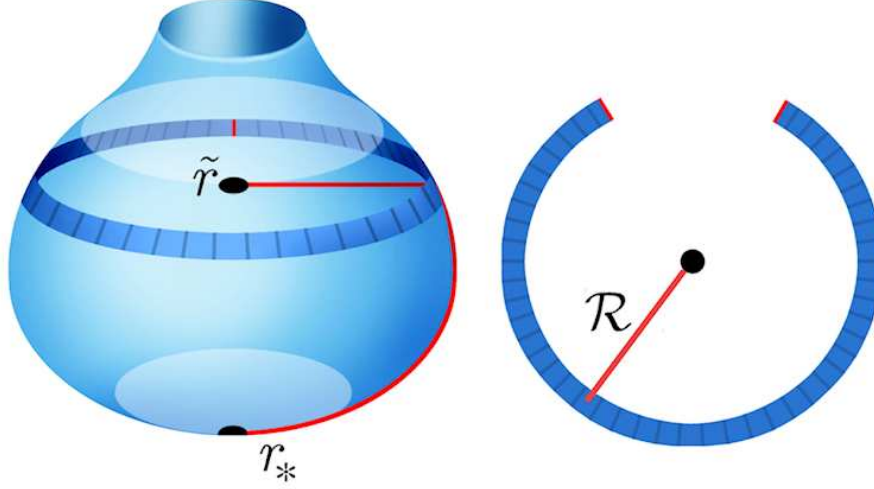


Figure 1. For a circle placed in a curved space (here on a curved 2-D surface), its geodesic radius r_* , circumferential radius \tilde{r} , and curvature radius \mathcal{R} are all different, $r_* \neq \tilde{r} \neq \mathcal{R}$.

space, they *uniquely* lead to the 3-D metric of the form,

$$ds^2 = \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4)$$

where r_0 is a constant. A choice $r_0 = 4M$ leads to correct values for both the perihelion advance and the light bending effects[‡].

2. The three radii of a circle

Consider a two dimensional curved, axisymmetric surface with the metric

$$ds^2 = dr_*^2 + [\tilde{r}(r_*)]^2 d\phi^2. \quad (5)$$

and a family of concentric circles $r_* = \text{const}$ in it. One of them is shown in Figure 2. Obviously, r_* is the geodesic radius and \tilde{r} is the circumferential radius of these circles,

$$(\text{geodesic radius}) \equiv \int_0^{r_*} ds_{|\phi=\text{const}} = r_*, \quad (6)$$

$$(\text{circumferential radius}) \equiv \frac{1}{2\pi} \int_0^{2\pi} ds_{|r_*=\text{const}} = \tilde{r}. \quad (7)$$

Let $\tau^i = \tilde{r}^{-1} \delta_\phi^i$ be a unit vector tangent to the circle. From the Frenet formula,

$$\frac{d\tau^i}{ds} = -\frac{1}{\mathcal{R}} \lambda^i, \text{ where } \lambda^i = (\text{unit normal to the circle}), \quad (8)$$

one deduces that the curvature radius \mathcal{R} may be defined by,

$$(\text{curvature radius}) \equiv \left[\frac{d\tau^i}{ds} \frac{d\tau_i}{ds} \right]^{-1/2} = \mathcal{R}. \quad (9)$$

[‡] Assuming that light moves along geodesic lines in space.

Two useful formulae for the curvature of the circle, $\mathcal{K} = 1/\mathcal{R}$, and for the Gaussian curvature \mathcal{G} of the surface with the metric (5) read,

$$\mathcal{K} = + \frac{1}{\tilde{r}} \left(\frac{d\tilde{r}}{dr_*} \right), \quad (10)$$

$$\mathcal{G} = - \frac{1}{\tilde{r}} \left(\frac{d^2\tilde{r}}{dr_*^2} \right). \quad (11)$$

Formula (10) follows from (9). For derivation of (11) see e.g. [2], Section 3.4.

3. Equations of motion

Let us consider a curve in space given by a parametric equation,

$$x^i = x^i(s), \quad (12)$$

where x^i are coordinates in space, and s is the length along the curve. If a body moves along this curve, its velocity equals,

$$v^i = \frac{dx^i}{dt} = \frac{ds}{dt} \frac{dx^i}{ds} = v\tau^i. \quad (13)$$

Here $v = ds/dt$ is the speed of the body and $\tau^i = dx^i/ds$ is a unit vector tangent to the curve (12), i.e. the direction of motion. The acceleration may be calculated as follow,

$$a^i = \frac{dv^i}{dt} = \frac{ds}{dt} \frac{d(v\tau^i)}{ds} = v^2 \left(\frac{d\tau^i}{ds} \right) + \tau^i v \frac{dv}{ds}. \quad (14)$$

Assuming circular motion with constant velocity, $v = \text{const}$, and applying (9) to calculate the term in brackets, we arrive at

$$a^i = v^2 \frac{1}{\mathcal{R}} \lambda^i, \quad (15)$$

which is the well known formula for the centrifugal acceleration.

Consider now circular motion around a spherically symmetric center of gravity. The Newtonian equation of motion, $F^i = ma^i$, takes the form,

$$- \nabla^i \Phi = v^2 \frac{1}{\mathcal{R}} \lambda^i, \quad (16)$$

where $F^i = -m\nabla^i \Phi$ is the gravitational force, and Φ is the gravitational potential. Three quantities characterize motion on a particular circular orbit: the angular velocity Ω , the angular speed v , and the specific angular momentum \mathcal{L} . They are related by,

$$v = \tilde{r}\Omega, \quad (17)$$

$$\mathcal{L} = \tilde{r}v = \tilde{r}^2\Omega. \quad (18)$$

Using (18), and multiplying its left side by λ_i , we transform the equation of motion (16) into a form which will be convenient later,

$$\lambda_i \nabla^i \Phi = \frac{\mathcal{L}^2}{\tilde{r}^2 \mathcal{R}}. \quad (19)$$

In this expression, λ^i is a unit, *outside pointing*, vector. Here “outside” has the absolute meaning — outside the center, in the direction towards infinity. We will calculate the left-hand side of this equation in the next Section.

4. Newton's gravity and Kepler's law

In an empty space, the gravitational potential Φ obeys the Laplace equation,

$$\nabla_i(\nabla^i\Phi) = 0. \quad (20)$$

Let us integrate (20) over the volume \mathbb{V} that is contained between two spheres, concentric with the gravity center, with sphere \mathbb{S}_1 being inside sphere \mathbb{S}_2 . We transform the volume integral into a surface integral, using the Gauss theorem

$$0 = \int_{\mathbb{V}} \nabla_i(\nabla^i\Phi) d\mathbb{V} = \int_{\mathbb{S}_1} (\nabla^i\Phi) N_i^{(1)} d\mathbb{S} + \int_{\mathbb{S}_2} (\nabla^i\Phi) N_i^{(2)} d\mathbb{S}. \quad (21)$$

The oriented surface elements on the spherical surfaces \mathbb{S}_1 and \mathbb{S}_2 may be written, respectively, as

$$N_i^{(1)} d\mathbb{S} = -\lambda_i d\mathbb{S}, \quad N_i^{(2)} d\mathbb{S} = +\lambda_i d\mathbb{S}, \quad (22)$$

therefore,

$$\int_{\mathbb{S}_1} (\nabla^i\Phi) \lambda_i d\mathbb{S} = \int_{\mathbb{S}_2} (\nabla^i\Phi) \lambda_i d\mathbb{S}. \quad (23)$$

This means that the value of the integral is the same, say S_0 , for all spheres around the gravity center. In addition, because of the spherical symmetry of the potential, the quantity $(\nabla^i\Phi) \lambda_i$ is constant over the sphere of integration. Thus,

$$S_0 = (\nabla^i\Phi) \lambda_i \int_{\mathbb{S}} d\mathbb{S} = 4\pi\tilde{r}^2 (\nabla^i\Phi) \lambda_i, \quad (\nabla^i\Phi) \lambda_i = \frac{S_0}{4\pi\tilde{r}^2} = \frac{GM}{\tilde{r}^2}. \quad (24)$$

Combining (24) with (19), we may finally write,

$$\mathcal{L}^2 = GM\mathcal{R}. \quad (25)$$

This is the Kepler Third Law. Using natural units for radius and frequency,

$$R_G = \frac{GM}{c^2} = M, \quad \Omega_G = \frac{c^3}{GM} = \frac{c}{M}, \quad (26)$$

we may write the formula for the Keplerian angular velocity as,

$$\left(\frac{\Omega}{\Omega_G} \right)^2 = R_G^3 \left(\frac{\mathcal{R}}{\tilde{r}^4} \right). \quad (27)$$

5. Epicyclic oscillations, the perihelion advance

Suppose that we slightly perturb a test-body on a circular orbit. This means that its angular momentum will not correspond to the Keplerian one, \mathcal{L}^2 , given by (25), but will be slightly different $\mathcal{L}^2 + \delta\mathcal{L}^2$. There will be also a small radial motion with velocity $(\delta\dot{r}_*)$ and acceleration $(\delta\ddot{r}_*)$. From (19) it follows that

$$\frac{GM}{\tilde{r}^2} - \frac{\mathcal{L}^2 + \delta\mathcal{L}^2}{\tilde{r}^2\mathcal{R}} = (\delta\ddot{r}_*). \quad (28)$$

Keeping the first order term in equation (28), and using

$$\delta\mathcal{L}^2 = \frac{d\mathcal{L}^2}{dr_*} (\delta r_*), \quad (29)$$

we arrive at the simple harmonic oscillator equation,

$$\omega^2(\delta r_*) + (\ddot{\delta r_*}) = 0, \quad (30)$$

where ω is the radial epicyclic frequency,

$$\omega^2 = \frac{1}{\tilde{r}^2 \mathcal{R}} \left(\frac{d\mathcal{L}^2}{dr_*} \right). \quad (31)$$

Using equations (25) and (26), we may write the expression for the epicyclic frequency in the form,

$$\left(\frac{\omega}{\Omega_G} \right)^2 = \left(\frac{d\mathcal{R}}{dr_*} \right) \frac{R_G^3}{\tilde{r}^2 \mathcal{R}}, \quad (32)$$

or comparing this with (27),

$$\left(\frac{\omega}{\Omega} \right)^2 = \left(\frac{d\mathcal{R}}{dr_*} \right) \frac{\tilde{r}^2}{\mathcal{R}^2} = \left(\frac{d\tilde{r}}{dr_*} \right)^2 - \tilde{r} \left(\frac{d^2 \tilde{r}}{dr_*^2} \right). \quad (33)$$

In a flat space, $r_* = \tilde{r} = \mathcal{R}$, and therefore $\omega = \Omega$, which implies that the slightly non-circular orbit is a closed curve, indeed an ellipse. In a curved space with $\mathcal{G} \neq 0$, one has $r_* \neq \tilde{r} \neq \mathcal{R}$, and consequently $\omega \neq \Omega$. The slightly non-circular orbit would not be a closed curve. It could be represented by a precessing ellipse, with two consecutive perihelia shifted by

$$\Delta\phi = T(\Omega - \omega) = 2\pi \left(1 - \frac{\omega}{\Omega} \right) = 2\pi \left[1 - \left(\frac{d\mathcal{R}}{d\tilde{r}} \frac{\tilde{r}^3}{\mathcal{R}^3} \right)^{1/2} \right], \quad (34)$$

where $T = 2\pi/\Omega$ is the orbital period.

6. A Newtonian experiment

Newtonian dynamics allows one to *measure* the circumferential \tilde{r} and curvature \mathcal{R} radii of circular orbits by measuring the gravitational a_G and centrifugal a_C radial accelerations for a circular orbit,

$$a_G = -\frac{GM}{\tilde{r}^2}, \quad a_C = \frac{V^2}{\mathcal{R}}. \quad (35)$$

In the Schwarzschild metric, the acceleration of a particle (a “planet”) moving with the orbital velocity v along a circular orbit equals,

$$a_i = \nabla_i \Psi + V^2 \frac{\nabla_i \mathbf{R}}{\mathbf{R}}. \quad (36)$$

Here $V = v/(1 - v^2)^{1/2}$, and the scalars Ψ and \mathbf{R} are expressed in terms of the time-symmetry Killing vector η^i , and the axial-symmetry Killing vector ξ^i ,

$$\Psi = -\frac{1}{2} \ln(\eta^i \eta_i), \quad \mathbf{R}^2 = -\frac{(\xi^k \xi_k)}{(\eta^i \eta_i)}. \quad (37)$$

In Schwarzschild coordinates this is, at the “equatorial plane” $\theta = \pi/2$,

$$(\eta^i \eta_i) = g_{tt} = 1 - \frac{2M}{r}, \quad (\xi^k \xi_k) = -r^2. \quad (38)$$

This allows one to calculate the results of the Newtonian experiment to measure the gravitational and centrifugal accelerations,

$$a_G = -\frac{1}{2} \frac{d}{dr} [\ln(1 - 2M/r)], \quad a_C = \frac{1}{2} V^2 \frac{d}{dr} \left[\ln \left(\frac{r^2}{1 - 2M/r} \right) \right]. \quad (39)$$

By comparing (35) and (39), one concludes that,

$$\tilde{r}(r) = r(1 - 2M/r)^{1/2}, \quad \mathcal{R}(r) = r \frac{1 - 2M/r}{1 - 3M/r}. \quad (40)$$

Note, that to $\mathcal{O}^1(M/r)$ accuracy this is $r = \tilde{r} = \mathcal{R}$. Therefore, curvature effects may appear at this order. We can also usefully calculate the derivative $dr_*(r)/dr$, as only the derivative, not the absolute value of $r_*(r)$ is of interest. The following equation follows from the definition of Frenet's curvature radius \mathcal{R}

$$\frac{dr_*}{dr} = \frac{\mathcal{R} d\tilde{r}}{\tilde{r} dr} = \frac{r - M}{r - 3M}. \quad (41)$$

The above formula allows one to write the metric of the 2-D space geometry, $ds^2 = dr_*^2 + \tilde{r}^2 d\phi^2$, measured in this Newtonian experiment,

$$ds^2 = \left(\frac{r - M}{r - 3M} \right)^2 dr^2 + r^2 \left(1 - \frac{2M}{r} \right) d\phi^2. \quad (42)$$

Inserting (40) into the Newtonian perihelion advance formula (34) one gets,

$$\frac{\Delta\phi}{2\pi} = 1 - \sqrt{1 + \frac{-6Mr^3 + 34M^2r^2 - 62M^3r + 36M^4}{r^4 - 5Mr^3 + 8M^2r^2 - 4M^3r}}. \quad (43)$$

Expanding this to the desired accuracy $\mathcal{O}^2(M/r)$, one finally gets the same value for the perihelion advance as calculated in Einstein's theory,

$$\Delta\phi = 6\pi \frac{M}{r} + \mathcal{O}^2 \left(\frac{M}{r} \right). \quad (44)$$

7. Light bending

Knowing the space geometry, given by equation (42), we may calculate the effect of *light bending* assuming that light travels along geodesic lines in space. In Newton's theory this assumption is equivalent to the *Fermat principle*, i.e. that light travels (with a constant speed) between two points A, B in space, minimizing the time travel T_{AB} . The equation of motion for the ϕ coordinate is, in these circumstances,

$$\frac{d^2\phi}{ds^2} + 2 \frac{r - M}{r(r - 2M)} \frac{dr}{ds} \frac{d\phi}{ds} = 0, \quad (45)$$

from which we find $\frac{d\phi}{ds}$ to be equal to

$$\frac{d\phi}{ds} = \frac{C}{r(r - 2M)}. \quad (46)$$

The integration constant can be evaluated at the perihelion location $r = R_0$ (i.e. where $d\phi/ds = 0$), yielding

$$\frac{d\phi}{ds} = \frac{\sqrt{R_0(R_0 - 2M)}}{r(r - 2M)}. \quad (47)$$

Using equations (42) and (47) we find also

$$\frac{dr}{ds} = \frac{r - 3M}{r - M} \sqrt{1 - \frac{R_0(R_0 - 2M)}{r(r - 2M)}}. \quad (48)$$

After dividing equation (47) by equation (48) and substituting $x = R_0/r$ the $d\phi/dr$ equation can be integrated from R_0 to ∞ (or x from 0 to 1), which will give us the half of $\pi + \delta$. Let us also define $\mu = M/R_0$, then

$$\frac{\pi + \delta}{2} = \int_0^1 \frac{1 - x\mu}{(1 - 2x\mu)(1 - 3x\mu)} \sqrt{\frac{1 - 2\mu}{1 - x^2(1 - 2\mu)/(1 - 2x\mu)}} dx. \quad (49)$$

This integration can be expanded in a Taylor series for μ :

$$\frac{\pi + \delta}{2} = \int_0^1 \frac{dx}{\sqrt{1 - x^2}} + \int_0^1 \frac{(3x^2 + 3x - 1)dx}{\sqrt{1 - x^2}(x + 1)} \mu + \mathcal{O}^2(\mu). \quad (50)$$

As the first component on the right hand side is equal to $\pi/2$, we conclude that

$$\delta \approx 2\mu \int_0^1 \frac{(3x^2 + 3x - 1)dx}{\sqrt{1 - x^2}(x + 1)} = 4 \frac{M}{R_0}. \quad (51)$$

Similar calculations in the Schwarzschild spacetime geometry give the same result§

$$\delta \approx 2\mu \int_0^1 \frac{1 - x^3}{(1 - x^2)^{3/2}} = 4 \frac{M}{R_0}. \quad (52)$$

Once again, the prediction of the Newtonian theory in the non-flat space is found to be consistent with observations (and with Einstein's general relativity).

8. Enhanced Newtonian Gravitational Theory

Jürgen Ehlers pointed out in 1961 that in Einstein's theory the curvature of the rest-space of irrotational matter is determined by its distribution and relative motion (see 1221 in his article [3]). The equations governing such 3-space curvature for arbitrary irrotational flows are given in [4]; see their equation (54). Consequently it makes sense to consider gravitational dynamics in the context of 3-dimensional curved Riemannian spaces. As Newtonian theory is an approximation to General Relativity Theory, it is therefore interesting to see what happens in the case of Newtonian theory in a curved 3-dimensional background space.

In the case of isometric flows, $\theta = \sigma_{ab} = 0$ and there is a potential such that $\dot{u}^a = U_{,a}$ where the gravitational potential U relates the Killing vector ξ to the unit 4-velocity u^a by $\xi^a = e^U u^a$ (see 1234 in Ehlers [3]). Then the relevant equation becomes

$${}^3R_{ab} = \tilde{\nabla}_a \tilde{\nabla}_b U + \tilde{\nabla}_a U \tilde{\nabla}_b U + \frac{2}{3} k \rho h_{ab}, \quad (53)$$

where $\tilde{\nabla}_a$ is the 3-dimensional covariant derivative, ρ is the energy density of matter, and we have assumed anisotropic stress is zero ($\pi_{ab} = 0$) and a vanishing cosmological

§ Which is twice the well-known *flat-space* and massive photon Newtonian prediction.

constant. Here $h_{ab} = g_{ab} - u_a u_b$ is the metric of the three-spaces orthogonal to u^a . This case will include static spherically symmetric spacetimes. Taking the trace of this equation gives (see equation (55) in [4])

$${}^3R = 2k\rho, \quad (54)$$

where the potential terms have gone because of the relation between the 3-dimensional and 4-dimensional covariant derivatives. Together with the Poisson equation and equation of motion it defines the *Enhanced Newtonian Gravitational Theory*,

$$\begin{aligned} {}^3g^{ik} \nabla_i \nabla_k \Phi &= -4\pi G\rho, \\ F_i &= ma_i, \\ {}^3R &= 2k\rho. \end{aligned} \quad (55)$$

For spherically symmetric spaces, the most general metric has the form,

$$ds^2 = A(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (56)$$

and in the vacuum case, $\rho = 0 = {}^3R$ one has

$${}^3R_{rr} = \frac{A'}{rA}, \quad (57)$$

$${}^3R_{\theta\theta} = \left(\frac{rA'}{2A^2} - \frac{1}{A} + 1 \right) \sin^2 \theta, \quad (58)$$

$${}^3R_{\phi\phi} = \frac{rA'}{2A^2} - \frac{1}{A} + 1, \quad (59)$$

$${}^3R = \frac{2}{r^2} - \frac{2}{Ar^2} + \frac{2A'}{A^2r} = 0. \quad (60)$$

Here the prime denotes a derivative with respect to r . Equation (60) has a unique solution,

$$A(r) = \left(1 - \frac{r_0}{r} \right)^{-1}, \quad (61)$$

with r_0 being an integration constant. Its value cannot be determined by equations (55), but instead must be chosen by correspondence with experiment^{||}. Using the same procedure as in Sections 6 and 7, one proves that the choice $r_0 = 4M$ gives the correct values for the perihelion advance and light bending (with accuracy $\mathcal{O}(r_0/r)$).

9. The two metrics

We have shown that “experimentally” established and the “theoretically” postulated Newtonian metrics of the curved 3-D space corresponding to a spherically symmetric body are, respectively,

$$ds^2 = \left(\frac{r-M}{r-3M} \right)^2 dr^2 + r^2 \left(1 - \frac{2M}{r} \right) (d\theta^2 + \sin^2 \theta d\phi^2),$$

^{||} In Einstein’s theory, when one derives the Schwarzschild metric, a constant of integration is determined in a similar way, i.e by correspondence with Newton’s theory.

$$(62)$$

$$ds^2 = \left(1 - \frac{4M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (63)$$

We have also shown that *any* spherically symmetric metric that obeys ${}^3R = 0$ must be isometric with (63). The Ricci scalar for the “experimental” metric may be calculated to be

$$\begin{aligned} ({}^3R)M^2 &= M^2 \frac{18(M/r)^3 - 10(M/r)^2}{r^2[4(M/r)^3 - 8(M/r)^2 + 5(M/r) - 1]} \\ &= 10(M/r)^4 + 32(M/r)^5 + \dots \\ &= 0 + \mathcal{O}^4(M/r). \end{aligned} \quad (64)$$

On the other hand a metric,

$$ds^2 = \frac{dr^2}{1 - \frac{4M}{r} + \alpha \left(\frac{M}{r}\right)^2 + \beta \left(\frac{M}{r}\right)^3 + \dots} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (65)$$

has the Ricci tensor,

$$({}^3R)M^2 = 2\alpha \left(\frac{M}{r}\right)^4 + 4\beta \left(\frac{M}{r}\right)^5 + \dots = 0 + \mathcal{O}^4(M/r). \quad (66)$$

Thus, the experimental metric (62) and the theoretical metric (63) describe, with accuracy $\mathcal{O}^2(M/r)$, *the same* geometry of space.

10. Conclusions

We demonstrated that a Newtonian physicist may experimentally determine the geometry of the 3-D space ${}^3g_{ik}^E$ by measuring gravitational and centrifugal accelerations. He may then predict by calculations the perihelion advance and the light bending as effects of the curvature of space. The predicted values agree with the ones measured. We also demonstrated that one may extend Newton’s theory of gravitation by adding an equation that links Ricci curvature of space with the density of matter. We calculated the resulting theoretical metric of space ${}^3g_{ik}^T$ assuming spherical symmetry. In this metric, the values of perihelion advance and light bending also agree with those observed. The two metrics represent the same geometry, ${}^3g_{ik}^E = {}^3g_{ik}^T$ with accuracy $\mathcal{O}^2(M/r)$.

Abramowicz [1] has shown that for spaces with constant Gaussian curvature Newton’s theory predicts no perihelion advance. We speculate that this is why Gauss (and other XIX century mathematicians) who might have calculated Newtonian orbits in curved spaces, would have missed the effect of perihelion advance. Most probably, they would calculate orbits in spaces with a constant Gaussian curvature first. Gauss almost certainly made this calculation. He was a master in calculating orbits. He made himself famous at the age of 23 by calculating the orbit of Ceres, discovered in 1801 by Piazzi. He seriously considered the possibility that our space is curved. He even attempted to determine the curvature of space by measuring angles in a big triangle (69 km, 84 km, 106 km) made by the summits of Brocken, Hoher Hagen and Großer Inselsberg. Gauss

was not quick in publishing his results concerning curved spaces. It is known that he discovered most of Bolyai's results, but never published them. Gauss died in February 1885, four years before Le Verrier discovered the effect of the perihelion of Mercury advance.

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